Lecture 12— Green's Functions

In lecture 10, we have briefly discussed how the Laplace equation generally has no solution, due to the fact one requires to prescribe Cauchy data u and $\partial_\nu u$ on $\partial\Omega$ simultaneously. This is of-course for the Cauchy problem. Now recall the generalized Green formula, where for any $u, \phi \in C^2(\Omega) \cap C^1(\overline{\Omega})$ with $\Phi(x) = E(x-y) + \phi(x)$

$$u(y) = \int_\Omega (u\Delta\phi + \Phi_y \Delta u) + \int_{\partial\Omega} (u\partial_\nu \Phi_y - \Phi_y \partial_\nu u). \tag{1}$$

Consider

$$\begin{cases} \Delta u = f & in\ \Omega \\ u = g & on\ \partial\Omega \end{cases} \tag{2}$$

otherwise known as the Dirichlet problem, and suppose ϕ_y satisfies

$$\begin{cases} \Delta\phi_y = 0 & \in \Omega \\ \phi_y = -E_y & \in \partial\Omega \end{cases} \tag{3}$$

we notice that term $\int_{\partial\Omega} \Phi_y \partial_\nu u$ in expression (1) vanishes since $\phi_y = -E_y$ on the boundary, hence $\partial_\nu u$ drops out from the expression which allows for the existence of a solution since we may now prescribe Cauchy data u alone on the boundary.

Definition 1. *We call a fundamental solution G(x,y) with pole y a $Green's\ function$ for the Dirichlet problem for the Laplace equation in the domain Ω if,*

$$G_y(x) \equiv G(x,y) := E(x-y) + \phi_y(x)$$

for $x \in \overline{\Omega},\ y \in \Omega,\ x \neq y$ where $\phi_y(x)$ satisfies (3)*.*

The solution for this (Dirichlet) problem is

$$u(y) = \int_\Omega G_y f + \int_{\partial\Omega} u\partial_\nu G_y \tag{4}$$

where $G_y(x)$ is a Green's Function for Ω.

One may think of this in the following manner.Suppose you want to construct a Green's function with property

$$\begin{cases} \Delta G = \delta & in\ \Omega \\ G = 0 & on\ \partial\Omega \end{cases}$$

It can be considered that this point as a point charge where one wants to solve for the potential field generated by it, in the free space case it would be E_y where our charge is located at y. Suppose now we want a solution to exhibit a potential distribution prescribed by g on a non-characteristic surface denoted $\partial\Omega$. It would be required to cancel the effect of the E_y on $\partial\Omega$ meanwhile remain the same elsewhere. This can be done by introducing an opposite charge to charge y such that the resultant

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distribution will cancel the effects on $\partial\Omega$. An example if $\partial\Omega$ is a plane, so we take the reflection of y with respect to plane $\partial\Omega$.

Does (4) solve (2) ?

$$\text{Yes, for nice domains } g \in C(\partial\Omega),\ f \in C^{0,\alpha}(\Omega).$$

$f = 0$ case :

$$\Delta u = 0 \ \in \Omega$$

$$u(x) \to g(z)\ as\ \Omega \ni x \to z \in \partial\Omega$$

Simple Observations

$$\frac{\partial G(x,y)}{\partial \nu(x)} = \frac{\partial[E(x-y) + \phi_y(x)]}{\partial \nu(x)} = \frac{\partial[E(x-y) + \phi_x(y)]}{\partial \nu(x)}$$

(nontrivial to prove this symmetry).

$$G(x,y) = G(y,x)$$

$$G\ is\ unique\ if\ it\ exists$$

$$G \le 0$$

$$G(x,y) \ge E(x-y)$$

Poisson Formula

Suppose we would like to solve the Dirichlet problem on a ball of radius R centred at the origin. We require to formulate a Greens function on the ball:

$$\Omega = B(0,R) = \{x:\ |x| < R\}$$

Take a point $y \in B_R$ and find the inverse with respect to the sphere. Simple analytical geometry reveals that the reflection y^* is

$$y^* = \frac{R^2}{|y|^2} y.$$

The boundary of Ω is in fact a sphere hence the set of all points on $\partial\Omega$ have a constant ratio between the distances to x from each of the points y and y^*; in particular if we define $|x-y| = r$ and $|x-y^*| = r_*$ then for all $x \in \partial\Omega$ ($x's$ on the sphere),

$$const = \frac{r_*^2}{r^2} = \frac{|x - \frac{R^2}{|y|^2} y|^2}{R^2 + |y|^2} \tag{5}$$

$$= \frac{|x|^2 + \frac{R^2}{|y|^2}}{R^2 + |y|^2} = \frac{R^2}{|y|^2} \frac{(|y|^2 + R^2)}{(R^2 + |y|^2)} = \frac{R^2}{|y|^2}. \tag{6}$$

Using the fundamental solution E_y, we have

$$E_y = \frac{1}{(2-n)|S^{n-1}|r^{2-n}}, \qquad E_{y^*} = \frac{1}{(2-n)|S^{n-1}|r_*^{2-n}} \qquad n > 2,$$

we can relate E_y and E_{y^*} for $x \in \partial\Omega$

$$\frac{E_{y^*}}{E_y} = \frac{r_*^{2-n}}{r^{2-n}} \implies E_{y^*} = \left(\frac{R}{|y|}\right)^{n-2} E_y$$

hence the expression

$$G_y = E_y - \left(\frac{|y|}{R}\right)^{2-n} E_{y^*}$$

vanishes when $x \in \partial\Omega$ by the relation above. Clearly, if we take $\phi_y = -\left(\frac{|y|}{R}\right)^{2-n} E_{y^*}$, the conditions in definition 1 hold hence G_y here is a Green function. Now suppose $u \in C^2(\overline{\Omega})$ and harmonic we have by the general Green's formula

$$u(y) = \int_{\Omega} \underbrace{(u\Delta E_{y^*} + G_y \Delta u)}_{=0} + \int_{\partial\Omega} (u\partial_\nu G_y - \underbrace{G_y \partial_\nu u}_{G_y|_{\partial\Omega}=0}).$$

hence our solution

$$u(y) = \int_{\partial\Omega} u\partial_\nu G_y$$

where $\partial_\nu G(x,y) = \frac{R^2-|y|^2}{R|S^{n-1}||x-y|^n}$, better known as the Poisson kernel

$$\implies u(y) = \frac{R^2 - |y|^2}{R|S^{n-1}|} \int_{|x|=R} \frac{u(x)}{|x-y|^n}\, dS_x, \qquad \forall y \in B_R$$

otherwise known as the Poisson Integral formula.

Using the same physical analogy with respect to potential fields above, the unit charge at y is Q then

$$Q^* = -\left(\frac{R}{|y|}\right)^{n-2} Q$$

hence $G(x,y) = \frac{c_n}{|x-y|^{n-2}} - \frac{c_n R^{n-2}}{|y|^{n-2}|x - \frac{R^2}{|y|^2} y|^{n-2}}$

$$\partial_\nu G(x,y) = \frac{R^2 - |y|^2}{R|S^{n-1}||x-y|^n}$$

so

$$u(y) = \frac{R^2 - |y|^2}{R|S^{n-1}|} \int_{\partial B_R} \frac{g(x)}{|x-y|^n} d^{n-1}x = \int_{\partial B_R} \pi(x,y) g(x) d^{n-1}x$$

with $\Delta u = 0$ in B_R. Take $z \in \partial B_R$

$$\pi(y,x) \geq 0, \qquad \pi(zt, x) \qquad take\ t \to 1$$

$$\int_{\partial B_R} \pi(y,x) d^{n-1}x = 1$$

$$\forall \delta > 0,\ \exists t^* \in (0,1)\ s.t\ \pi(zt,x) \leq \delta\ for\ \|x-z\|_2 \geq \delta,\ t > t^*.$$

Alternative pf of Koebe's Converse

Suppose $u \in C(\Omega)$ satisfies MVP in Ω. Take an arbitrary ball centred at $y \in \Omega$ with $\overline{B_R(y)} \subset \Omega$. Solve

$$\begin{cases} \Delta v = 0 \ in \ B_R(y) \\ v = u \ on \ \partial B_R(y) \end{cases}$$

$w = u - v$ satisfies the MVP in $B_R(y)$ and $w = 0$ on $\partial B_R(y)$

$$\implies w \equiv 0 \in B_R(y)$$

$$\implies u = v \qquad \Delta u = 0$$

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